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# Target Space Duality II: Applications\*

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## Abstract

We apply the framework developed in Target Space Duality I: General Theory. We show that both nonabelian duality and Poisson-Lie duality are examples of the general theory. We propose how the formalism leads to a systematic study of duality by studying few scenarios that lead to open questions in the theory of Lie algebras. We present evidence that there are probably new examples of irreducible target space duality.

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# 1 Introduction

In this article we consider some applications of the general theory derived in article I [1]. We show that nonabelian duality [2, 3, 4, 5, 6, 7, 8, 9, 10] and Poisson-Lie duality [11, 12, 13, 14] are special cases of the general theory. In fact we show that nonabelian duality is a special case of a more general situation. The spirit of this paper is that there are natural geometric scenarios that need to be explored. We explore a few of the easier ones and see that they lead to open mathematical questions in the theory of Lie algebras. For example, Lie bialgebras and generalizations, and  $R$ -matrices naturally appear in this framework. References to equations and sections in article I are preceded by I, *e.g.*, (I-8.3).

## 2 Examples with a flat $\psi$ connection

### 2.1 General remarks

To best understand how to use the equations given in the Section I-8.1 it is best to do a few examples. The examples we will consider in Section 2 assume that the connection  $\psi_{ij}$  is flat. We are mostly interested in local properties so we might as well assume  $P$  is parallelizable. We can use parallel transport with respect to this connection to get a global framing. In this special framing the connection coefficients vanish and thus we can make the substitution  $\psi_{ij} = 0$  in all the equations in Section I-8.1. A previous remark tells us that in this case  $f_{ijk}$  and  $\tilde{f}_{ijk}$  are pullbacks respectively of tensors on  $M$  and  $\widetilde{M}$ . Consequently we have that  $df_{ijk} = f'_{ijkl}\theta^l$  and  $d\tilde{f}_{ijk} = \tilde{f}''_{ijkl}\tilde{\theta}^l$ , *i.e.*,  $f''_{ijkl} = \tilde{f}''_{ijkl} = 0$ . Because we are interested in mostly local considerations we might as well assume both  $M$  and  $\widetilde{M}$  are parallelizable.

### 2.2 The tensor $n_{ij}$ is the pullback of a tensor on $\widetilde{M}$

Here we assume that the connection  $\psi$  is flat. In Section I-8.2 we considered what happened if  $n_{ij}$  was covariantly constant in this section we relax this condition to one where  $n_{ij}$  only depends on a natural subset of the variables. We assume that  $n_{ij}$  is the pullback of a tensor on  $\widetilde{M}$ . This means that

$$dn_{ij} = n''_{ijk}\tilde{\theta}^k \quad \text{or equivalently} \quad n'_{ijk} = 0. \quad (2.1)$$

An equivalent formulation is that  $\frac{1}{2}n_{ij}\tilde{\theta}^i \wedge \tilde{\theta}^j$  is the pullback of a 2-form on  $\widetilde{M}$ . We immediately see from (I-8.17) and (I-8.18) that

$$n''_{ijk} = m_{kl}f_{lij}, \quad (2.2)$$

$$n''_{kij} - n''_{kji} = m_{jl}f_{lki} - m_{il}f_{lkj} = \tilde{f}_{ij}m_{lk}. \quad (2.3)$$

A brief computation shows that  $d(m_{ji}\tilde{\theta}^j) = 0$  and therefore we can locally find  $n$  functions  $p_1, \dots, p_n$  such that

$$dp_i = m_{ji}\tilde{\theta}^j. \quad (2.4)$$

Note that since  $m_{ij}$  is invertible the differentials  $\{dp_1, \dots, dp_n\}$  are linearly independent. The functions  $p_1, \dots, p_n$  are the pullbacks of functions locally defined on  $\widetilde{M}$  and they define a local coordinate system on  $\widetilde{M}$ . We immediately see that

$$dn_{ij} = f_{kij}dp_k. \quad (2.5)$$

The integrability condition  $0 = d^2n_{ij} = f'_{kijl}\theta^l \wedge dp_k$  immediately tells us that  $f'_{kijl} = 0$  since  $\{\theta_1, \dots, \theta_n, dp_1, \dots, dp_n\}$  are linearly independent. Since  $f'_{ijk} = f''_{ijk} = 0$  we see that  $f_{ijk}$  are constants and thus  $M$  is diffeomorphic to a Lie group  $G$ . The  $\theta^i$  are the pullbacks by  $\Pi$  of the left invariant Maurer-Cartan forms on  $G$ . The next question is whether the pullback metric  $\theta^i \otimes \theta^i$  is invariant under the action of the group. Choose a  $G$  left-invariant vector field  $X$  “along”  $M$ , *i.e.*,  $\iota_X\tilde{\theta}^i = 0$ . A brief computation shows that

$$\mathcal{L}_X(\theta^i \otimes \theta^i) = X^k f_{ijk}(\theta^i \otimes \theta^j + \theta^j \otimes \theta^i) \quad (2.6)$$

The metric on  $M$  is  $G$ -invariant if and only if  $f_{ijk} = -f_{jik}$ , *i.e.*, the structure constants are totally antisymmetric, a standard result.

We can integrate (2.5) to obtain

$$n_{ij} = n_{ij}^0 + f_{kij}p_k \quad (2.7)$$

where  $n_{ij}^0$  are constants. Define the constant tensor  $m^0$  by  $m_{ij}^0 = \delta_{ij} + n_{ij}^0$ . Equation (2.3) immediately gives us an expression for  $\tilde{f}_{ijk}$  in terms of  $f_{ijk}$  and  $p_l$ . It is a straightforward computation to verify that  $\tilde{f}_{ijk}$  satisfies the integrability conditions for  $d\tilde{\theta}^l = -\frac{1}{2}\tilde{f}_{lij}\tilde{\theta}^i \wedge \tilde{\theta}^j$ . In general  $\tilde{f}_{ijk}$  are not constants and are thus not generically Maurer-Cartan forms for a Lie group.

To determine  $H$  we use (I-8.15). Note that our hypotheses imply that the left hand side is automatically zero. After using the Jacobi identity satisfied by the  $f_{ijk}$  we find that

$$H_{ijk} = f_{lij}n_{lk}^0 + f_{ljk}n_{li}^0 + f_{lki}n_{lj}^0. \quad (2.8)$$

If we define a left invariant 2-form by  $n^0 = \frac{1}{2}n_{ij}^0\theta^i \wedge \theta^j$  then the above equation is  $H = -dn^0$  which tells us that  $H$  is cohomologically trivial. We use (I-8.16) to determine  $\tilde{H}$ :

$$\begin{aligned}\tilde{H}_{ijk} &= (\tilde{f}_{ijk} + \tilde{f}_{jki} + \tilde{f}_{kij}) \\ &+ H_{ijk} - (f_{ijk} + f_{jki} + f_{kij}) .\end{aligned}\tag{2.9}$$

### 2.2.1 Cotangent bundle duality

We will see in this section that cotangent bundle duality is a special case of what we have been discussing in Section 2.2. In particular this corresponds to what is often called “nonabelian duality”. The cotangent bundle has a natural symplectic structure and thus we automatically have a candidate symplectic manifold  $P$  for free. Assume the manifold  $M$  is parallelizable. This means that the cotangent bundle  $T^*M$  is trivial, *i.e.*, it is a product space,  $P = T^*M = M \times \mathbb{R}^n$ . The projections  $\Pi$  and  $\tilde{\Pi}$  will be taken to be the cartesian projections and therefore  $\tilde{M} = \mathbb{R}^n$ . If  $\theta^i$  is an orthonormal coframe on  $M$  then the canonical 1-form on  $T^*M$  may be written as  $\alpha = p_i\theta^i$ . The  $p_i$  are coordinates along the fibers of  $\Pi$ . The fibers of  $\tilde{\Pi}$  (diffeomorphic to  $M$ ) given by  $dp_i = 0$ ,  $i = 1, \dots, n$  are the same as the fibers given by  $\tilde{\theta}^i = 0$ ,  $i = 1, \dots, n$  therefore there must exist an invertible matrix defined by functions  $\hat{m}_{ij}$  such that

$$dp_j = \hat{m}_{ij}\tilde{\theta}^i .\tag{2.10}$$

The canonical symplectic form is given by  $\beta = d\alpha = dp_j \wedge \theta^j - \frac{1}{2}p_k f_{kij}\theta^i \wedge \theta^j$ . In going from  $\beta$  to  $\gamma$  the  $\tilde{\theta} \wedge \theta$  term is not changed and so we immediately learn that  $\hat{m}_{ij} = m_{ij} = \delta_{ij} + n_{ij}$ . Taking the exterior derivative of (2.10) leads to  $n'_{ijk} = 0$  and (2.3). Thus  $n_{ij}$  is the pullback of a tensor on  $\mathbb{R}^n$  and we are back to our general discussion given in Section 2.2. This is an example of what is often called “nonabelian duality” which has generating function given by (I-2.2) with  $\alpha = p_k\theta^k$ . We know that  $M$  is a Lie group  $G$ ,  $T^*G = G \times \mathfrak{g}^*$  and thus  $\tilde{M} = \mathfrak{g}^*$ . The metric on  $\mathfrak{g}^*$  is immediately computable from (2.10) since  $m_{ij} = \delta_{ij} + n_{ij}^0 + p_k f_{kij}$ . It is worth remarking that because  $\tilde{M} = \mathfrak{g}^* = \mathbb{R}^n$  there is an abelian Lie group action on  $\tilde{M}$  which does not leave the metric invariant. Said differently the  $dp$  are the Maurer-Cartan forms for the abelian Lie group  $\mathfrak{g}^*$ . This statement is made in anticipation of our discussion about Poisson-Lie duality in Section 3.

“Nonabelian duality” usually refers to the case with  $n_{ij}^0 = 0$  where (2.8) tells us that  $H_{ijk} = 0$ . We can compute  $\tilde{H}$  from (2.9) directly or it was already remarked in Section I-7.2 that  $\tilde{B}_{ij}$  is easily determined.

### 2.2.2 Cotangent bundle duality and gauge invariance

In this section we revisit cotangent bundle duality and try to understand geometrically the role of the  $B$  field gauge transformation. We assume the manifold  $M$  has a trivial cotangent bundle  $T^*M = M \times \mathbb{R}^n$  with canonical 1-form  $\alpha = p_i \theta^i$  and symplectic form  $\beta = d\alpha$ . We modify the discussion of Section 2.2.1 by demanding that the bifibration not be given by the cartesian projections. We want the vertical fibers to be original ones so we have  $\Pi : (x, p) \in T^*M \mapsto x \in M$ . On the other hand the projection  $\tilde{\Pi} : T^*M \rightarrow \tilde{M}$  will not be the canonical projection. The fibers of this projection are “slanted” relative to the fibers of the cartesian projection  $\tilde{\Pi}_c$ . Note that the fibers of  $\tilde{\Pi}$  are the integral manifolds of the Pfaffian equations  $\tilde{\theta}^i = 0$ . From general principles we know that

$$dp_i = \hat{m}_{ji} \tilde{\theta}^j + u_{ji} \theta^j \quad (2.11)$$

for some functions  $\hat{m}, u$ . Geometrically this is the statement that the fibers of  $\tilde{\Pi}$  are slanted relative to the fibers of  $\tilde{\Pi}_c$ . As before the structure of the symplectic form leads to the result that  $\hat{m}_{ij} = m_{ij}$ . The integrability condition  $d^2 p_i = 0$  leads to the following equations

$$n''_{ijk} - n''_{ikj} = \tilde{f}_{ljk} m_{li} , \quad (2.12)$$

$$u'_{jik} - u'_{kij} = -f_{ljk} u_{li} , \quad (2.13)$$

$$u''_{jik} = n'_{kij} . \quad (2.14)$$

Comparing (2.12) with (I-8.18) we see that  $n'_{ijk} = 0$  and thus  $n_{ij}$  is the pullback of a tensor on  $\tilde{M}$ . The general discussion of Section 2.2 tells us that there exists function  $\hat{p}_i$  in  $T^*M$  such that  $d\hat{p}_i = m_{ji} \tilde{\theta}^j$ . Geometrically this is just the statement that the fibers is given by  $\hat{p}_i = \text{constant}$ . Note that (2.14) tells us that  $u''_{ijk} = 0$  and thus  $u_{ij}$  is the pullback of a tensor on  $M$ . Equation (2.11) tells us that  $d(p_i - \hat{p}_i) = u_{ji} \theta^j$  and thus we see that  $p_i = \hat{p}_i + k_i$  where  $k_i$  are pullbacks of functions on  $M$ , *i.e.*,  $k_i = k_i(x)$ . Thus we see that the canonical transformation generated by  $\int p_i \theta^i$  is gauge equivalent to the one generated by  $\int \hat{p}_i \theta^i$  and corresponds to a different choice of fibration.

### 2.2.3 Can $\tilde{M}$ naturally be a Lie group?

Is it possible for the dual manifold to be *naturally* a Lie group under the assumptions underlying the discussions in Section 2.2? By “naturally” we mean that the  $\tilde{\theta}^i$  are the Maurer-Cartan forms for a Lie group  $\tilde{G}$ . Solving (2.3) for  $\tilde{f}_{ijk}$  generally leads to a non-constant solution. There is a possibility that the solution may be constant, *i.e.*,

$(\widetilde{M}, \widetilde{g}, \widetilde{B})$  is naturally a Lie group. Unfortunately we will see that both  $\mathfrak{g}$  and  $\widetilde{\mathfrak{g}}$  must be abelian and so there are no new interesting examples. Inserting (2.7) into (2.3) and using the linear independence of the  $dp_i$  leads to two equations:

$$m_{jl}^0 f_{ki}^l - m_{il}^0 f_{kj}^l = \widetilde{f}_{ij}^l m_{lk}^0, \quad (2.15)$$

$$f_{il}^m f_{jk}^l = f_{il}^m \widetilde{f}_{jk}^l. \quad (2.16)$$

To obtain the latter equation we used the Jacobi identity satisfied by  $f_{ijk}$ . The indices have also been set at their natural (co)variances for future convenience. The  $\widetilde{f}_{jk}^i$  satisfy the Jacobi identity if they satisfy the two equations above because of our remark about integrability after (2.7). You should think of  $M$  as a Lie group  $G$  with Lie algebra  $\mathfrak{g}$  and similarly for  $\widetilde{M}$ . Let  $\text{ad}_X(Y) = [X, Y]$  be the adjoint action of  $\mathfrak{g}$ . Let  $[\cdot, \cdot]^\sim$  be the Lie bracket on  $\widetilde{\mathfrak{g}}$  with adjoint action denoted by  $\widetilde{\text{ad}}_{\widetilde{X}} \widetilde{Y} = [\widetilde{X}, \widetilde{Y}]^\sim$ . The reduction of the structure group of  $P$  to  $O(n)$  meant that we could identify the horizontal tangent space with the vertical tangent space and thus we can identify  $\mathfrak{g}$  with  $\widetilde{\mathfrak{g}}$ . We should think of a single vector space  $V$  with two Lie brackets giving us two Lie algebras  $\mathfrak{g} = (V, [\cdot, \cdot])$  and  $\widetilde{\mathfrak{g}} = (V, [\cdot, \cdot]^\sim)$ . In this notation (2.16) becomes

$$\text{ad}_X(\text{ad}_Y - \widetilde{\text{ad}}_Y) = 0. \quad (2.17)$$

There are some immediate important consequences of this equation. Let  $\mathfrak{d}$  be the vector subspace of  $\mathfrak{g}$  spanned by  $\text{ad}_Y Z - \widetilde{\text{ad}}_Y Z$  for all  $Y, Z$ . The subspace  $\mathfrak{d}$  is contained in the center  $\mathfrak{z}$  of  $\mathfrak{g}$  by (2.17). If  $\mathfrak{d} \neq 0$ , *i.e.*,  $\text{ad} \neq \widetilde{\text{ad}}$ , then the center  $\mathfrak{z}$  is a nontrivial abelian ideal in the Lie algebra  $\mathfrak{g}$  and thus  $\mathfrak{g}$  is not semisimple. If  $\mathfrak{d} = 0$  then we have that  $\text{ad} = \widetilde{\text{ad}}$  and we will show that  $\mathfrak{g}$  is abelian when we incorporate (2.15) into our reasoning. Note that if  $\mathfrak{z} = 0$  then  $\mathfrak{d} = 0$ .

Let us study the implications of  $\text{ad} = \widetilde{\text{ad}}$ , *i.e.*,  $\mathfrak{d} = 0$ . Equation (2.15) may be rewritten as

$$2f_{kij} = (f_{ijk} + f_{jki} + f_{kij}) + (n_{il}^0 f_{ljk} + n_{jl}^0 f_{lki} + n_{kl}^0 f_{lij}). \quad (2.18)$$

Notice that the right hand side is totally antisymmetric under permutations of  $i, j, k$ . This means that  $f_{kij}$  is totally antisymmetric and thus the metric  $\theta^i \otimes \theta^i$  is a bi-invariant positive definite metric on  $G$ . The proposition proven in Appendix A tells us that there is a decomposition into ideals  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a}$  where  $\mathfrak{k}$  is a compact semisimple Lie algebra and  $\mathfrak{a}$  is an abelian Lie algebra. The next part of the argument only involves the compact ideal  $\mathfrak{k}$  and without any loss we assume  $\mathfrak{a} = 0$  for the moment. Using the antisymmetry of the structure constants the above may be rewritten as

$$-f_{ijk} = n_{il}^0 f_{ljk} + n_{jl}^0 f_{lki} + n_{kl}^0 f_{lij}.$$

Using the remark made immediately after (2.8) we see that  $\frac{1}{3!}f_{ijk}\theta^i \wedge \theta^j \wedge \theta^k = -dn^0$ . The closed 3-form  $\frac{1}{3!}f_{ijk}\theta^i \wedge \theta^j \wedge \theta^k$  is exact and this contradicts  $H^3(\mathfrak{k}) \neq 0$  as discussed in Appendix A. If  $\text{ad} = \widetilde{\text{ad}}$  then we conclude that  $\mathfrak{g} = \mathfrak{a}$  is abelian and we are back in familiar territory.

The next we consider the general case by exploiting the observation that  $\mathfrak{d} \subset \mathfrak{z}$ . Since we have a positive definite metric on  $\mathfrak{g}$  there is an orthogonal direct sum decomposition  $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{z}^\perp$ . The orthogonal complement  $\mathfrak{z}^\perp$  is not generally an ideal because the metric is not necessarily  $(\text{ad } \mathfrak{g})$ -invariant. Nevertheless we can choose an orthonormal basis  $\{e_\alpha\}$  for  $\mathfrak{z}$  and an orthonormal basis  $\{e_a\}$  for  $\mathfrak{z}^\perp$ . Greek indices are associated with  $\mathfrak{z}$ , indices in  $\{a, b, c, d\}$  are associated with  $\mathfrak{z}^\perp$ ; and indices in  $\{i, j, k, l\}$  run from  $1, \dots, \dim \mathfrak{g}$  and are associated with all of  $\mathfrak{g}$ . The Lie algebra  $\mathfrak{g}$  is given by

$$[e_\alpha, e_j] = 0, \quad (2.19)$$

$$[e_a, e_b] = f_{ab}^c e_c + f_{ab}^\gamma e_\gamma. \quad (2.20)$$

Note that  $\mathfrak{h} = \mathfrak{g}/\mathfrak{z}$  is a Lie algebra since  $\mathfrak{z}$  is an ideal. It follows that the structure constants of  $\mathfrak{h}$  are  $f_{ab}^c$ . Also,  $\mathfrak{g}$  is a central extension of  $\mathfrak{h}$  with extension cocycle  $f_{ab}^\gamma$ . It is well known that the cocycle is trivial if  $f_{ab}^\gamma = t_{ab}^\gamma f_{ab}^c$  corresponding to a redefinition of the basis given by  $e_a \rightarrow e_a + t_{ab}^\gamma e_\gamma$ . The Lie algebra  $\tilde{\mathfrak{g}}$  must have the form below because the two Lie brackets are the same modulo the center  $\mathfrak{z}$ :

$$[e_\alpha, e_j]^\sim = \tilde{f}_{\alpha j}^\gamma e_\gamma, \quad (2.21)$$

$$[e_a, e_b]^\sim = f_{ab}^c e_c + \tilde{f}_{ab}^\gamma e_\gamma. \quad (2.22)$$

In equation (2.15) choose  $k = \gamma$  then the left hand side vanishes and the equation becomes  $0 = \tilde{f}_{ij}^l m_{l\gamma}^0 = \tilde{f}_{ij}^d m_{d\gamma}^0 + \tilde{f}_{ij}^\delta m_{\delta\gamma}^0$ . Using the different choices for  $(i, j)$  we find

$$\begin{aligned} \tilde{f}_{\alpha j}^\delta &= -\tilde{f}_{\alpha j}^d m_{d\gamma}^0 ((m^0)^{-1})^{\gamma\delta} = 0, \\ \tilde{f}_{ab}^\delta &= -f_{ab}^d m_{d\gamma}^0 ((m^0)^{-1})^{\gamma\delta}. \end{aligned} \quad (2.23)$$

These equations tell us that  $\tilde{\mathfrak{g}}$  is a central extension of  $\mathfrak{h}$  with a trivial cocycle. Next choose  $i = \alpha$  in (2.15) with result  $0 = m_{\alpha l}^0 f_{kj}^l$ . Choosing  $k = a$  and  $j = b$  leads to

$$f_{ab}^\gamma = -((m^0)^{-1})^{\gamma\alpha} m_{\alpha c}^0 f_{ab}^c = 0 \quad (2.24)$$

which tells us that the cocycle  $f_{ab}^\gamma$  is also trivial. Finally choose  $(i, j, k) = (a, b, c)$  in (2.15) and substitute (2.23) and (2.24) for  $\tilde{f}_{ab}^\delta$  and  $f_{ab}^\gamma$  to obtain

$$\begin{aligned} (m_{bd}^0 - m_{b\gamma}^0 ((m^0)^{-1})^{\gamma\alpha} m_{\alpha d}^0) f_{ca}^d &= (m_{ad}^0 - m_{a\gamma}^0 ((m^0)^{-1})^{\gamma\alpha} m_{\alpha d}^0) f_{cb}^d \\ &= f_{ab}^d (m_{dc}^0 - m_{d\alpha}^0 ((m^0)^{-1})^{\alpha\gamma} m_{\gamma c}^0). \end{aligned} \quad (2.25)$$

Next we observe that since  $m_{\alpha\beta}^0 = \delta_{\alpha\beta} + n_{\alpha\beta}$  we conclude that  $((m^0)^{-1})^{\alpha\beta} = s^{\alpha\beta} + a^{\alpha\beta}$  where  $s^{\alpha\beta}$  is symmetric and positive definite and  $a^{\alpha\beta}$  is skew. In particular note that  $-m_{b\gamma}^0((m^0)^{-1})^{\gamma\alpha}m_{\alpha d}^0 = s^{\gamma\alpha}m_{\gamma b}^0m_{\alpha d}^0 + a^{\gamma\alpha}m_{\gamma b}^0m_{\alpha d}^0$  where the first term is symmetric and positive definite and the second is skew. Using this we immediately see that

$$m_{bd}^0 - m_{b\gamma}^0((m^0)^{-1})^{\gamma\alpha}m_{\alpha d}^0 = S_{bd} + N_{bd}$$

where  $S_{bd}$  is symmetric and positive definite and  $N_{bd}$  is skew. We can use  $S_{ab}$  as a second metric on  $\mathfrak{g}$  and use it to “raise and lower” the indices in (2.25) leading to

$$2f_{cab} = (f_{abc} + f_{bca} + f_{cab}) + (N_{ad}f_{bc}^d + N_{bd}f_{ca}^d + N_{cd}f_{ab}^d). \quad (2.26)$$

This tells us that  $f_{abc}$  is totally antisymmetric and therefore  $S_{ab}$  is an invariant metric on  $\mathfrak{h}$ . The same chain of arguments used after (2.18) tell us that  $\mathfrak{h}$  is abelian which implies that  $\tilde{f}_{ab}^\delta = 0$  and  $f_{ab}^\gamma = 0$  concluding that  $\mathfrak{g}$  and  $\tilde{\mathfrak{g}}$  are abelian Lie algebras.

#### 2.2.4 Connection with $R$ -matrices

Equation (2.15) is closely related to the theory of  $R$ -matrices developed by Semenov-Tian-Shansky [15] which is different than the one developed by Drinfeld [16]. The discussion here suggests that Poisson-Lie groups may play a role in duality, see Section 3. We observe that (2.15) may be rewritten as

$$(m^0)_{jl}^{-1}f_{kil} - (m^0)_{kl}^{-1}f_{jil} = (m^0)_{jl}^{-1}(m^0)_{km}^{-1}\tilde{f}_{nlm}m_{ni}^0. \quad (2.27)$$

Next we show that this equation describes a potential double Lie algebra structure on  $\mathfrak{g}$  à la Semenov-Tian-Shansky (B.11). Assume the Lie algebra  $\mathfrak{g}$  admits an invariant metric  $K$ . For example if the Lie algebra is semisimple then  $K$  may be taken to be the Killing metric. Let us use the indices  $a, b, c, d, e$  to denote generic components in a generic basis. In terms of a basis  $(e_1, \dots, e_n)$  the structure constants are given by  $[e_a, e_b] = f_{ab}^c e_c$ . For the moment it is best to forget about the orthonormal basis we were previously using before because the metric  $K$  may not be related to the previous metric. The components of the invariant metric are given by  $K_{ab} = K(e_a, e_b)$ . We will use  $K$  to identify  $\mathfrak{g}$  with  $\mathfrak{g}^*$ , *i.e.*, raise and lower indices. The tensor with components  $K^{ab}$  is the inverse of the invariant metric, *i.e.*, the induced metric on  $\mathfrak{g}^*$ . The statement that  $K$  is  $\mathfrak{g}$ -invariant is equivalent to  $f_{abc}$  being totally antisymmetric. With these assumptions equation (B.13) may be rewritten as

$$(R^d_e K^{ea})f_{cd}^b - (R^d_e K^{eb})f_{cd}^a = K^{aa'}K^{bb'}K_{cc'}(f_{a'b'}^c)_R. \quad (2.28)$$

The indices of  $R$  and  $f$  have their natural (co)variances.



We are now going to compare (2.28) with (2.27). Remember that (2.27) is valid in an orthonormal basis with respect to a specific metric on  $\mathfrak{g}$  which may not be related to the invariant metric  $K$ . What we will have to do is express (2.28) in the orthonormal basis but this will be simple because we adjusted all the (co)variances correctly in the equation above. The indices  $i, j, k, l, m, n$  refer to our orthonormal basis. We raise and lower indices using the Kronecker delta tensor. The only potential confusion is that we have to be careful and remember that  $K_{ab}$  becomes  $K_{ij}$  and the inverse invariant metric  $K^{ab}$  becomes  $K_{ij}^{-1}$ . The correspondence is made by choosing  $R$  (in our orthonormal basis) to be defined by

$$(m^0)_{ij}^{-1} = R_{jl} K_{li}^{-1} . \quad (2.29)$$

The  $R$ -matrices we are considering are invertible because both  $m^0$  and  $K$  are always invertible. This is different that in the Drinfeld case where the  $R$ -matrix is skew adjoint and may not be invertible. If you think of  $K$  as a map  $K : \mathfrak{g} \rightarrow \mathfrak{g}^*$  then the equation above is  $(m^0)^t = K R^{-1} : \mathfrak{g} \rightarrow \mathfrak{g}^*$  which suggests that  $m^0$  should be interpreted as a map  $m^0 : \mathfrak{g}^* \rightarrow \mathfrak{g}$ . Comparing the right hand sides of (2.27) and (2.28) we see that

$$\tilde{f}_{ilm} R_{lj} R_{mk} = R_{il} (f_{ljk})_R . \quad (2.30)$$

If  $R_{ij}$  satisfies the modified Yang Baxter equation (B.10) then  $(f^i_{jk})_R$  are the structure constants of a Lie algebra  $\mathfrak{g}_R$  associated with a Lie group  $G_R$ . Let  $(\mu^1, \dots, \mu^n)$  be the left invariant Maurer-Cartan forms for  $G_R$ :

$$d\mu_l = -\frac{1}{2} (f_{ljk})_R \mu_j \wedge \mu_k .$$

Going to a different basis given by  $\lambda_i = R_{il} \mu_l$  we see that

$$d\lambda_i = -\frac{1}{2} \tilde{f}_{ilm} \lambda_l \wedge \lambda_m .$$

We conclude that if  $R$  is a solution of the modified classical Yang Baxter equation then we can construct the Lie algebra  $\tilde{\mathfrak{g}}$  with structure constant  $\tilde{f}_{ijk}$  and an associated Lie group  $\tilde{G}$ .

In our situation we also have to impose a second equation (2.16) and this is very restrictive. Our in-depth analysis that eliminated all except the abelian case. Logically, there is the possibility of non-trivial solutions by the use of  $R$ -matrices. The reason is that using (2.30) and (B.11) we can rewrite (2.16) as

$$0 = f^m_{il} (f^l_{jk} R^j_p R^k_q - f^r_{nq} R^n_p R^l_r - f^r_{pn} R^n_q R^l_r) .$$

The stuff between the parentheses is the left hand side of modified classical Yang Baxter equation (B.12). If  $R$  satisfies the modified classical Yang Baxter equation the above becomes  $0 = -c f^m_{il} f^l_{pq}$ . The solution to this equation is  $c = 0$  or  $\text{ad}_X \text{ad}_Y = 0$ . We know from our general analysis that it is unnecessary to proceed along these lines.

## 2.3 Symmetric duality

In the previous sections we had  $\alpha = p_i \theta^i$  for some functions  $p_i$  on  $P$ . A consequence was that the  $p_i$  were good coordinates on the fibers of  $\Pi$  and these fibers were also lagrangian submanifolds of  $\beta$ . There was a certain asymmetry in the way fibers of  $\Pi$  and of  $\tilde{\Pi}$  were treated. In this section we consider a more symmetric situation. To motivate the ensuing presentation let us go to the case of  $M = \mathbb{R}^n$  and  $P = T^*M$  where we have the symplectic form  $\beta = dp_i \wedge dq^i$ . Up to constant 1-forms the most general antiderivative is  $\alpha = (1 - u)p_i dq^i - u q^i dp_i$  where  $u \in \mathbb{R}$  is a constant.

Let  $U \subset P$  be a neighborhood, we can always write

$$\alpha = \tilde{q}_i \theta^i - q^i \tilde{\theta}_i, \quad (2.31)$$

where  $q^i$  and  $\tilde{q}_i$  are local functions on  $U$ . We now make some special assumptions about the functions  $q$  and  $\tilde{q}$ . Assume there exist matrix valued functions  $E$  and  $\tilde{E}$  such that

$$dq^i = E_{ij} \theta^j, \quad (2.32)$$

$$d\tilde{q}_j = \tilde{\theta}_i \tilde{E}_{ij}. \quad (2.33)$$

These functions are not arbitrary and must satisfy a variety of constraints. For example, the ranks are constrained by equation (2.41) below. For example if  $E$  and  $\tilde{E}$  are invertible then the functions  $(q, \tilde{q})$  are independent in the sense that the map  $(q^1, \dots, q^n, \tilde{q}_1, \dots, \tilde{q}_n) : U \rightarrow \mathbb{R}^{2n}$  is of rank  $2n$ . Consequently the fibers of  $\Pi$  are locally described by  $(q = \text{constant})$  and the fibers of  $\tilde{\Pi}$  by  $(\tilde{q} = \text{constant})$ . In general  $E$  or  $\tilde{E}$  will not be invertible. In this case not all the functions will be independent and  $(q = \text{constant})$  will define a family of manifolds each diffeomorphic to  $\widetilde{M}$ .

As before we use the prime and double prime notation to denote derivatives in the appropriate directions. The equation  $d^2 q^i = 0$  tells us that

$$E''_{ijk} = 0, \quad (2.34)$$

$$E'_{ijk} - E'_{ikj} = -E_{il} f_{ljk}. \quad (2.35)$$

Likewise the equation  $d^2 \tilde{q} = 0$  tells us that

$$\tilde{E}'_{ijk} = 0, \quad (2.36)$$

$$\tilde{E}''_{kjl} - \tilde{E}''_{ljk} = -\tilde{f}_{ikl} \tilde{E}_{ij}. \quad (2.37)$$

Note that you can solve (2.35) for  $E'_{ijk}$  as a function of  $E_{il} f_{ljk}$  and you can solve (2.37) for  $\tilde{E}''_{ijk}$  and as function of  $\tilde{f}_{ikl} \tilde{E}_{ij}$ .

Next we compute  $\beta = d\alpha$ :

$$\beta = (\tilde{E}_{ij} + E_{ij})\tilde{\theta}^i \wedge \theta^j - \frac{1}{2}\tilde{q}_i f_{ijk}\theta^j \wedge \theta^k + \frac{1}{2}q^i \tilde{f}_{ijk}\tilde{\theta}^j \wedge \tilde{\theta}^k . \quad (2.38)$$

In general neither the fibers of  $\Pi$  or of  $\tilde{\Pi}$  are lagrangian submanifolds of  $\beta$ . This is very different from the cotangent bundle cases previously discussed. Next we make we write  $E$  and  $\tilde{E}$  as

$$E_{ij} = \sigma_{ij} + \nu_{ij} , \quad (2.39)$$

$$\tilde{E}_{ij} = \tilde{\sigma}_{ij} + \tilde{\nu}_{ij} \quad (2.40)$$

where  $\sigma$  and  $\tilde{\sigma}$  are symmetric, and  $\nu$  and  $\tilde{\nu}$  are antisymmetric. Comparing to (I-6.5) we see that

$$\delta_{ij} = \sigma_{ij} + \tilde{\sigma}_{ij} \quad \text{and} \quad n_{ij} = \nu_{ij} + \tilde{\nu}_{ij} . \quad (2.41)$$

The matrix valued functions  $E$  and  $\tilde{E}$  cannot be arbitrary and must satisfy conditions given by the above. We remark that  $n'_{ijk} = m'_{ijk} = E'_{ijk} + \tilde{E}'_{ijk} = E'_{ijk} = \sigma'_{ijk} + \nu'_{ijk}$ . Since  $n_{ij} = -n_{ji}$  we immediately see that

$$\sigma'_{ijk} = 0 , \quad \text{and} \quad E'_{ijk} = n'_{ijk} = \nu'_{ijk} . \quad (2.42)$$

Also  $E''_{ijk} = 0$  and thus we see that  $\sigma''_{ijk} = 0$  and  $\nu''_{ijk} = 0$ . This means that  $\sigma_{ij}$  is constant. Similarly we conclude that  $\tilde{\sigma}_{ij}$  is constant,  $\tilde{E}''_{ijk} = n''_{ijk} = \tilde{\nu}''_{ijk}$  and  $\tilde{\nu}'_{ijk} = 0$ .

We can take the results given above and insert into (I-8.18) and (I-8.17) to obtain

$$E'_{ijk} = n'_{ijk} = \nu'_{ijk} = -\tilde{f}_{lij}E_{lk} . \quad (2.43)$$

$$\tilde{E}''_{ijk} = n''_{ijk} = \tilde{\nu}''_{ijk} = +\tilde{E}_{kl}f_{lij} . \quad (2.44)$$

We now insert the above into (2.35) and (2.37) to obtain the basic equations

$$\tilde{E}_{kl}f_{lij} - \tilde{E}_{jl}f_{lik} = \tilde{f}_{ljk}\tilde{E}_{li} , \quad (2.45)$$

$$\tilde{f}_{lij}E_{lk} - \tilde{f}_{lik}E_{lj} = E_{il}f_{ljk} . \quad (2.46)$$

Note that

$$dE_{ij} = E'_{ijk}\theta^k = \tilde{f}_{lij}E_{lk}\theta^k = \tilde{f}_{lij}dq^l \quad (2.47)$$

$$d\tilde{E}_{ij} = \tilde{E}''_{ijk}\tilde{\theta}^k = \tilde{E}_{kl}f_{lij}\tilde{\theta}^k = f_{lij}d\tilde{q}^l \quad (2.48)$$

where we used (2.32) and (2.33). We know that  $df_{lij} = f'_{lij}\theta^k$  and  $d\tilde{f}_{lij} = \tilde{f}'_{lij}\tilde{\theta}^k$ . Therefore by taking the exterior derivatives of (2.47) and (2.48) we learn that

$$\tilde{E}_{ml}(df_{lij}) = 0 , \quad (2.49)$$

$$(d\tilde{f}_{lij})E_{lm} = 0 . \quad (2.50)$$

To make progress we have to make some assumptions. The simplest assumption is that  $E_{ij} = 0$ . In this case we are back to the discussion given in Section 2.2.1. In this paragraph we use the notation that a Lie group  $G$  via nonabelian duality is dual to  $\tilde{G} \approx \mathfrak{g}^*$ . You can generalize cotangent bundle duality along the following lines. Consider matrices with the same  $2 \times 2$  block form

$$E = \begin{pmatrix} E^{(1)} & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{E} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{E}^{(2)} \end{pmatrix}.$$

This will lead to a manifold  $M = \tilde{G}^{(1)} \times G^{(2)}$  and  $\tilde{M} = G^{(1)} \times \tilde{G}^{(2)}$ .

Next we look at the case where  $\tilde{E}_{ij}$  is invertible. Equation (2.49) tells us that  $f_{ijk}$  are constants and thus  $M$  is naturally a Lie group  $G$  with structure constants  $f_{ijk}$ . We can immediately integrate (2.48) obtaining

$$\tilde{E}_{ij} = \tilde{\sigma}_{ij} + \tilde{\nu}_{ij}^0 + f_{kij} \tilde{q}^k, \quad (2.51)$$

where  $\tilde{\nu}^0$  is a constant tensor. With this information we can use (2.45) to determine  $\tilde{f}_{ijk}$ . It is straightforward to verify that  $\tilde{f}_{ijk}$  satisfy the integrability conditions for (I-8.2) with  $\psi_{ij} = 0$ . All we have to do is to find a tensor  $E_{ij}$  that satisfies (2.46) and (2.47). Note that (2.41) tells us that  $\sigma_{ij} = \delta_{ij} - \tilde{\sigma}_{ij}$ . This case merits further analysis.

If besides  $\tilde{E}_{ij}$  being invertible we also impose that  $E_{ij}$  is invertible then  $\tilde{f}_{ijk}$  are constant (see (2.49)) and  $\tilde{M}$  is naturally a Lie group  $\tilde{G}$ . We can integrate (2.47) to obtain

$$E_{ij} = \sigma_{ij} + \nu_{ij}^0 + \tilde{f}_{lij} q^l, \quad (2.52)$$

where  $\nu^0$  is a constant tensor. It is convenient to define

$$E^0 = \sigma + \nu^0, \quad (2.53)$$

$$\tilde{E}^0 = \tilde{\sigma} + \tilde{\nu}^0. \quad (2.54)$$

We can insert (2.51) and (2.52) into (2.45) and (2.46) and expand both sides in powers of  $q$  and  $\tilde{q}$  to obtain:

$$\tilde{E}_{jl}^0 f_{ki}^l - \tilde{E}_{il}^0 f_{kj}^l = \tilde{f}_{ij}^l \tilde{E}_{lk}^0, \quad (2.55)$$

$$f_{il}^m f_{jk}^l = f_{il}^m \tilde{f}_{jk}^l, \quad (2.56)$$

$$\tilde{f}_{ki}^l E_{lj}^0 - \tilde{f}_{kj}^l E_{li}^0 = E_{kl}^0 f_{ij}^l, \quad (2.57)$$

$$\tilde{f}_{il}^m \tilde{f}_{jk}^l = \tilde{f}_{il}^m f_{jk}^l, \quad (2.58)$$

where the Jacobi identity was used to simplify the above. We are now in a situation very similar to that in Section 2.2.3. The difference is that the relevant metrics are

now  $\sigma_{ij}$  and  $\tilde{\sigma}_{ij}$ . The difficulty arises in that we have lost positive definiteness of the metrics. The only constraint is that  $\sigma + \tilde{\sigma} = I$ . If either  $\sigma$  or  $\tilde{\sigma}$  is definite then an analysis along the lines of Section 2.2.3 leads to the conclusion that  $\mathfrak{g}$  and  $\tilde{\mathfrak{g}}$  are abelian. If both are indefinite or both are singular then the analysis previously provided breaks down. This situation also merits further study.

### 3 Poisson-Lie duality

Here we discuss a beautiful example of scenario I-3 described at the end of Section I-8.1 where we are given a special symplectic bifibration and we have to construct the metrics and antisymmetric tensors on  $M$  and  $\widetilde{M}$ . The Drinfeld double Lie group is an example of a special symplectic bifibration. The metrics and antisymmetric tensors constructed in this manner correspond to the Poisson-Lie duality of Klimcik and Severa [11]. The explicit duality transformation was obtained by Sfetsos [14]. We can use our general framework to determine both by making educated guesses. The Drinfeld double  $G_D$  is a Lie group whose Lie algebra  $\mathfrak{g}_D$  is a Lie bialgebra, see Appendix B.1. The bifibration is by Lie groups  $G$  and  $\tilde{G}$  with respective Lie algebras  $\mathfrak{g}$  and  $\tilde{\mathfrak{g}} \approx \mathfrak{g}^*$ . The Lie algebras are related by  $\mathfrak{g}_D = \mathfrak{g} \oplus \tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}^*$ . If  $\{T_a\}$  is a basis for  $\mathfrak{g}$  and  $\{\tilde{T}^a\}$  is the associated dual basis for  $\mathfrak{g}^*$  then

$$\begin{aligned} [T_a, T_b] &= C^c_{ab} T_c, \\ [T_a, \tilde{T}^b] &= \tilde{C}^{bc}_a T_c - C^b_{ac} \tilde{T}^c, \\ [\tilde{T}^a, \tilde{T}^b] &= \tilde{C}^{ab}_c \tilde{T}^c, \end{aligned}$$

where  $C^c_{ab}$  and  $\tilde{C}^{bc}_a$  are respectively the structure constants for  $G$  and  $\tilde{G}$ . The two sets of structure constants must satisfy compatibility condition (B.4). To write down the symplectic structure in a convenient way we introduce some notation slightly different than the one given in [11]. Let  $g \in G$  then the adjoint representation on  $\mathfrak{g}$  is given by  $gT_b g^{-1} = T_a a^a_b(g)$ . One also has  $g\tilde{T}^a g^{-1} = a^a_b(g^{-1})(\tilde{T}^b + \Pi^{bc}(g)T_c)$  where  $\Pi^{ab} = -\Pi^{ba}$ . Similarly one has that if  $\tilde{g} \in \tilde{G}$  then  $\tilde{g}\tilde{T}^a \tilde{g}^{-1} = \tilde{T}^a \tilde{a}_a^b(\tilde{g})$  and  $\tilde{g}T_a \tilde{g}^{-1} = \tilde{a}_a^b(\tilde{g}^{-1})(T_b + \tilde{\Pi}_{bc}(\tilde{g})\tilde{T}^c)$  where  $\tilde{\Pi}$  is antisymmetric. It is worthwhile to note that if  $g = e^{x^a T_a}$  then  $\Pi^{ab}(g) = x^c \tilde{C}^{ab}_c + O(x^2)$  and similarly for  $\tilde{\Pi}_{ab}$ . Drinfeld shows that the bivector  $\Pi^{ab} T_a \wedge T_b$  on  $G$  defines a Poisson bracket that is compatible with the group multiplication law [16, 17]. A Poisson-Lie group is a Lie group with a Poisson structure which is compatible with the group operation. Thus we have that  $G$  is a Poisson-Lie group. Note that the Poisson bivector is degenerate. Similarly the bivector  $\tilde{\Pi}_{ab} \tilde{T}^a \wedge \tilde{T}^b$  makes  $\tilde{G}$  a Poisson Lie group. Klimcik and Severa discovered that the sigma model defined

on the Poisson-Lie group  $G$  is dual to the sigma model defined on the Poisson-Lie group  $\tilde{G}$  hence the name Poisson-Lie duality. To exhibit the duality transformation we write down the symplectic structure on  $G_D$ . We note that the Drinfeld double is not a Poisson-Lie group in the Poisson structure associated with the symplectic structure since the Poisson bivector would be nondegenerate. In the (perfect) Drinfeld double every element  $k \in G_D$  can uniquely be written as  $k = gu$  or  $k = vh$  where  $g, h \in G$  and  $u, v \in \tilde{G}$ . The inverse function theorem shows that you can choose  $h$  and  $u$  as local coordinates on  $G_D$  near the identity. Let  $\lambda = h^{-1}dh$  and  $\tilde{\lambda} = u^{-1}du$  be respectively the left invariant Maurer-Cartan forms on  $G$  and  $\tilde{G}$ . The symplectic structure [18, 19] may be written as

$$\begin{aligned} 2\beta &= (du u^{-1})_a \wedge (g^{-1}dg)^a + (v^{-1}dv)_a \wedge (dh h^{-1})^a, \\ &= \tilde{\lambda}_a \wedge \left[ \lambda^b + \tilde{\lambda}_c \Pi^{cb}(h^{-1}) \right] (I - \tilde{\Pi}(u^{-1})\Pi(h^{-1}))^{-1}{}_b{}^a \\ &\quad + \left[ \tilde{\lambda}_b + \lambda^c \tilde{\Pi}_{cb}(u^{-1}) \right] \wedge \lambda^a (I - \Pi(h^{-1})\tilde{\Pi}(u^{-1}))^{-1}{}_a{}^b. \end{aligned} \quad (3.1)$$

Using  $\beta$  we can construct the duality transformations. All we have to verify is that we get metrics and antisymmetric tensor fields on  $G$  and  $\tilde{G}$ .

Klimcik and Severa show that the symmetric and antisymmetric parts of the rank two tensors  $E = \lambda^t \otimes (F^{-1} + \Pi)^{-1} \lambda$  and  $\tilde{E} = \tilde{\lambda}^t \otimes (F + \tilde{\Pi})^{-1} \tilde{\lambda}$  are respectively the metrics and antisymmetric tensors for the dual sigma models on  $G$  and  $\tilde{G}$ . The coefficients  $F_{ab}$  are constants. By an appropriate choice of basis for  $\mathfrak{g}$  one can always choose  $F = I + b$  where  $b$  is antisymmetric. For pedagogical reasons we first look at the special case where  $b = 0$  which is analogous to choosing  $n^0 = 0$  in (2.7). By looking at (3.1) we make an educated guess and conjecture that our orthonormal bases should be given by  $\theta$  in  $G$  and  $\tilde{\theta}$  in  $\tilde{G}$  where

$$\lambda = (I + \Pi)\theta \quad \text{and} \quad \tilde{\lambda} = (I + \tilde{\Pi})\tilde{\theta}. \quad (3.2)$$

We can verify that this is in agreement with the Klimcik and Severa data by noting that  $E = \theta^t \otimes (I - \Pi)\theta$  and  $\tilde{E} = \tilde{\theta}^t \otimes (I - \tilde{\Pi})\tilde{\theta}$ . The symmetric parts on  $E$  and  $\tilde{E}$  in this basis are respectively  $\theta^t \otimes \theta$  and  $\tilde{\theta}^t \otimes \tilde{\theta}$  and thus we see that we have orthonormal bases on  $G$  and  $\tilde{G}$ . Likewise we see that the components of the antisymmetric tensors in this basis are given by  $-\Pi$  and  $-\tilde{\Pi}$  respectively. Note that  $\theta$  ( $\tilde{\theta}$ ) is well defined on  $G$  ( $\tilde{G}$ ) because  $\Pi$  ( $\tilde{\Pi}$ ) is defined on  $G$  ( $\tilde{G}$ ).

We are now ready to verify that there is a duality transformation. Postulate that frames given by  $\theta$  and  $\tilde{\theta}$  are the orthonormal ones we need. Rewrite (3.1) in the orthonormal frame where you find

$$m = (I - \tilde{\Pi})(I - \Pi\tilde{\Pi})^{-1}(I + \Pi), \quad (3.3)$$

$$\tilde{l} = -(I - \tilde{\Pi})\Pi(I - \tilde{\Pi}\Pi)^{-1}(I + \tilde{\Pi}), \quad (3.4)$$

$$l = -(I - \Pi)\tilde{\Pi}(I - \Pi\tilde{\Pi})^{-1}(I + \Pi), \quad (3.5)$$

using the notation in (I-3.5). A brief computation shows that  $m = I + n$  where  $n$  is antisymmetric and given by

$$\begin{aligned} n &= \sum_{k=1}^{\infty} \left[ (\Pi\tilde{\Pi})^k - (\tilde{\Pi}\Pi)^k \right] \\ &+ \sum_{k=0}^{\infty} (\Pi\tilde{\Pi})^k \Pi - \sum_{k=0}^{\infty} \tilde{\Pi}(\Pi\tilde{\Pi})^k. \end{aligned}$$

In this frame  $m$  is already in normal form and we can proceed. Note that  $\tilde{n} = -n$  as follows from (I-3.15). Using (I-3.8) and (I-3.9) we see that  $B = l - m + I = -\Pi$  and  $\tilde{B} = \tilde{l} + m - I = -\tilde{\Pi}$ . The important result here is that  $B$  and  $\tilde{B}$  are quantities which respectively live on  $G$  and  $\tilde{G}$  and thus we have constructed the Poisson-Lie duality of Klimcik and Severa for the special case  $b = 0$ .

The general solution for arbitrary  $b$  is given by choosing the orthonormal frames to be given by

$$\lambda = (I + \Pi F)\theta \quad \text{and} \quad \tilde{\lambda} = (F + \tilde{\Pi})\tilde{\theta}. \quad (3.6)$$

We have  $E = \theta^t \otimes (F - F^t \Pi F)\theta$  and  $\tilde{E} = \tilde{\theta} \otimes (F^t - \tilde{\Pi})\tilde{\theta}$  with the components of the antisymmetric tensor fields given by  $B = b - F^t \Pi F$  and  $\tilde{B} = -b - \tilde{\Pi}$  in this basis. The components of the symplectic form in this basis are

$$m = (F^t - \tilde{\Pi})(I - \Pi\tilde{\Pi})^{-1}(I + \Pi F), \quad (3.7)$$

$$\tilde{l} = -(F^t - \tilde{\Pi})\Pi(I - \tilde{\Pi}\Pi)^{-1}(F + \tilde{\Pi}), \quad (3.8)$$

$$l = -(I - F^t \Pi)\tilde{\Pi}(I - \Pi\tilde{\Pi})^{-1}(I + \Pi F), \quad (3.9)$$

A brief computation shows that  $m = I + n$  where  $n$  is antisymmetric and given by

$$\begin{aligned} n &= -b + \sum_{k=1}^{\infty} \left[ F^t (\Pi\tilde{\Pi})^k - (\tilde{\Pi}\Pi)^k F \right] \\ &+ \sum_{k=0}^{\infty} F^t (\Pi\tilde{\Pi})^k \Pi F - \sum_{k=0}^{\infty} \tilde{\Pi}(\Pi\tilde{\Pi})^k. \end{aligned}$$

Using (I-3.8) and (I-3.9) we see that  $B = l - m + I = b - F^t \Pi F$  and  $\tilde{B} = \tilde{l} + m - I = -b - \tilde{\Pi}$ . We have reproduced the ansatz of Klimcik and Severa and the duality transformation of Sfetsos.

## 4 Infinitesimally homogeneous $n_{ij}$

### 4.1 General theory

In this section we address the question, “What if  $n_{ij}$  is the same everywhere?” We will see that this is a much weaker condition than saying  $n' = n'' = 0$  which we already studied in Section I-8.2 and lead to abelian duality. We will show that  $P$  is a homogeneous space under certain assumptions. First we have to address the question of what does “same everywhere” mean. The best way to do this is to exploit some ideas developed by Singer [20] for the study of homogeneous spaces. It is convenient to work in the bundle  $\mathcal{F}(P)$  of the adapted orthogonal frames we have been using. This bundle has structure group  $O(n)$  and it admits a global coframing given by  $(\theta^i, \tilde{\theta}^j, \psi_{kl})$  where  $(\theta^i, \tilde{\theta}^j)$  are the canonical 1-forms on the frame bundle and  $\psi_{ij}$  is an  $O(n)$  connection. Remember that the Maurer-Cartan form on  $O(n)$  is the restriction of  $\psi$  to a fiber of  $\mathcal{F}(P)$ . The relationships among the geometries of  $M, \widetilde{M}, P$  are encapsulated in the Cartan structural equations for  $\mathcal{F}(P)$ :

$$d\theta^i = -\psi_{ij} \wedge \theta^j - \frac{1}{2} f_{ijk} \theta^j \wedge \theta^k, \quad (4.1)$$

$$d\tilde{\theta}^i = -\psi_{ij} \wedge \tilde{\theta}^j - \frac{1}{2} \tilde{f}_{ijk} \tilde{\theta}^j \wedge \tilde{\theta}^k, \quad (4.2)$$

$$d\psi_{ij} = -\psi_{ik} \wedge \psi_{kj} - T''_{ijlm} \theta^l \wedge \tilde{\theta}^m. \quad (4.3)$$

where  $f_{ijk} = -f_{ikj}$ ,  $\tilde{f}_{ijk} = -\tilde{f}_{ikj}$  and  $T''_{ijkl} = -T''_{jikl}$ . Here  $f, \tilde{f}, T''$  are all functions on  $\mathcal{F}(P)$ . Note that there is torsion arising from the reduction of the structure group. The ideal generated by  $\{\theta^1, \dots, \theta^n\}$  is a differential ideal with integral submanifolds being the restriction of  $\mathcal{F}(P)$  to the fibers of  $\Pi$ . The ideal generated by  $\{\tilde{\theta}^1, \dots, \tilde{\theta}^n\}$  is a differential ideal with integral submanifolds being the restriction of  $\mathcal{F}(P)$  to the fibers of  $\tilde{\Pi}$ . The degenerate quadratic forms  $\theta^i \otimes \theta^i$  and  $\tilde{\theta}^i \otimes \tilde{\theta}^i$  on  $\mathcal{F}(P)$  are respectively pullbacks of the metrics on  $M$  and  $\widetilde{M}$ . The pullback of the Riemannian connection on the frame bundle of  $M$  is schematically  $\psi + f\theta$  and likewise for  $\widetilde{M}$ . Said differently, when restricting  $\psi_{ij}$  to a “horizontal fiber” you get an orthogonal connection on the fiber with torsion, *etc.* We remind the reader that a tensor in  $P$  is a collection of functions on  $\mathcal{F}(P)$  which transform linearly under the action of  $O(n)$  on  $\mathcal{F}(P)$ , *i.e.*, as you change frames the “tensor” transforms appropriately. For future use the frame dual to the coframe  $(\theta^i, \tilde{\theta}^j, \psi_{kl})$  will be denoted by  $(e_i, \tilde{e}_j, E_{kl})$ . The horizontal vector fields with respect to  $\psi$  are spanned by  $\{e_A\} = \{e_i, \tilde{e}_j\}$ .

We are now ready to define the statement “ $n_{ij}$  is the same everywhere”. Pick a point  $b \in \mathcal{F}(P)$ . If we go to a rotated frame  $Rb$ ,  $R \in O(n)$ , then  $n_{ij}(b)$  becomes  $n_{ij}(Rb) =$



$R_{ik}R_{jl}n_{kl}(b)$ . Notice that as we move along the fiber going through  $b$  we will get the full orbit of  $n_{ij}(b)$  under  $O(n)$ . Thus to make sense of “ $n_{ij}$  is the same everywhere” we should not really talk about  $n_{ij}$  but about the invariants of antisymmetric tensors under the orthogonal group. We should be thinking in terms of the space of orbits of antisymmetric tensors under  $O(n)$ . In the frame bundle, the functions  $n_{ij}$  define a map  $n : \mathcal{F}(P) \rightarrow \bigwedge^2(\mathbb{R}^n)$ . We say that  $P$  is *n-homogeneous* if the image of the map  $n : \mathcal{F}(P) \rightarrow \bigwedge^2(\mathbb{R}^n)$  is a single  $O(n)$ -orbit. Said differently, you get the same  $2 \times 2$  block diagonalization of  $n_{ij}$  at each point of  $P$ . This is a weaker condition than covariantly constant  $n$ . If  $P$  is simply connected then a covariantly constant  $n_{ij}$  is determined by parallel transporting  $n_{ij}$  from a reference point. The value of  $n$  at the reference point determines  $n$  everywhere.

The condition that  $P$  be *n-homogeneous* is not strong enough for us. This leads to the notion of “infinitesimally *n-homogeneous*” where not only is  $n$  the same everywhere but also the first  $(N + 1)$  covariant derivatives of  $n$ . Let  $\nabla n$  denote the covariant derivative of  $n$ ,  $\nabla^2 n$  the second covariant derivative of  $n$ , etc. Consider the map

$$\rho^s = (n, \nabla n, \nabla^2 n, \dots, \nabla^s n) : \mathcal{F}(P) \rightarrow \mathbb{R}^{J_s},$$

where  $J_s$  is an integer we do not compute. We say that  $P$  is *infinitesimally n-homogeneous* if image of the map  $\rho^{N+1}$  is a single  $O(n)$ -orbit. The integer  $N$  is determined inductively as follows.

First we do a rough argument and afterwards we state Singer’s result. Look at  $\rho^0 = n : \mathcal{F}(P) \rightarrow \bigwedge^2(\mathbb{R}^n)$  and pick a point  $n^0$  in the orbit. Consider  $\mathcal{B} = \{b \in \mathcal{F}(P) : n(b) = n^0\}$ . Note that  $\mathcal{B}$  is a sub-bundle of  $\mathcal{F}(P)$  because  $n(\mathcal{F}(P))$  is a single orbit. If  $K'_0 \subset O(n)$  is the isotropy group of  $n^0$  then the action of  $K'_0$  on a point  $b \in \mathcal{B}$  leaves you in  $\mathcal{B}$ . Thus  $\mathcal{B}$  is a principal sub-bundle of  $\mathcal{F}(P)$  with structure group  $K'_0$ . The choice of  $n^0$  has broken the symmetry group to  $K'_0$ . Now let us be precise about Singer’s result. Pick a  $b_0 \in \mathcal{F}(P)$ . There exists a principal sub-bundle  $\mathcal{B}_0 \subset \mathcal{F}(P)$  containing  $b_0$  such that  $n$  is constant on  $\mathcal{B}_0$  and the structure group  $K_0 \subset O(n)$  of  $\mathcal{B}_0$  is the connected component of the isotropy group of  $n(b_0)$ . Note that for a generic orbit,  $K_0$  will be a maximal torus of  $O(n)$ .

Next we invoke  $\nabla n$  to reduce the symmetry group some more. We use  $\rho^1$  and apply Singer’s theorem to it. There exists a principal sub-bundle  $\mathcal{B}_1 \subset \mathcal{B}_0$  containing  $b_0$  such that  $(n, \nabla n)$  is constant on  $\mathcal{B}_1$  and the structure group  $K_1 \subset K_0$  of  $\mathcal{B}_1$  is the connected component of the isotropy group of  $(n(b_0), \nabla n(b_0))$ . We continue the procedure by looking at  $\rho^2, \rho^3, \dots$  and finding a sequence of principal sub-bundles  $\mathcal{F}(P) \supset \mathcal{B}_0 \supset \mathcal{B}_1 \supset \dots \supset \mathcal{B}_N \supset \mathcal{B}_{N+1}$  with respective structure groups  $O(n) \supset K_0 \supset K_1 \supset \dots \supset K_N \supset K_{N+1}$ . Since  $O(n)$  is finite dimensional there exists an integer  $N$  such that the

chain of groups satisfies the property that  $K_0 \neq K_1 \neq \dots \neq K_{N-1} \neq K_N = K_{N+1}$ . In fact Singer establishes that  $\mathcal{B}_N = \mathcal{B}_{N+1}$ , henceforth denoted by  $H$ , is a principal bundle with structure group  $K = K_N$ . Our arguments show that  $\rho^{N+1} = (n, \nabla n, \dots, \nabla^{N+1} n)$  is constant on  $H$ . Note that structure group  $K$  is the connected component of the isotropy group of  $\rho^N(b_0)$ . The chain of groups tells us that  $N \leq \frac{1}{2}n(n-1)$  and later on we will see that we also require  $N \geq 1$ .

Next we show that  $H$  is a Lie group and conclude that  $P = H/K$  is a homogeneous space. The strategy is to write down the Cartan structural equations for the principal bundle  $H$  and show that they are actually the Maurer-Cartan equations for a group. Pick a point  $b \in \mathcal{F}(P)$  and observe that  $d\rho^N$  is tangent to the orbit because  $\rho^N(\mathcal{F}(P))$  is a single orbit. The orbit is generated by the action of  $O(n)$  therefore for  $X \in T_b\mathcal{F}(P)$  there exists a linear map  $\Xi : T_b\mathcal{F}(P) \rightarrow \mathfrak{so}(n)$  such that  $d\rho^N(X) = \Xi(X) \cdot \rho^N(b)$ . Use the standard metric on  $SO(n)$  to write an orthogonal decomposition  $\mathfrak{so}(n) = \mathfrak{k} \oplus \mathfrak{k}^\perp$  where  $\mathfrak{k}$  is the Lie algebra of  $K$  and  $\mathfrak{k}^\perp$  is its orthogonal complement. Let  $b_0 \in H$  then we observe that if  $\Xi, \Xi'$  are such that for  $X \in T_{b_0}\mathcal{F}(P)$  you have  $d\rho^N(X) = \Xi(X) \cdot \rho^N(b_0) = \Xi'(X) \cdot \rho^N(b_0)$  then  $\Xi'(X) - \Xi(X) \in \mathfrak{k}$ . At  $b_0 \in H$  you can uniquely specify  $\Xi$  by requiring that  $\Xi(X) \in \mathfrak{k}^\perp$ . We will make this choice. Note that we allow  $X$  to be in the full tangent space  $T_{b_0}\mathcal{F}(P)$ . In summary, for  $b_0 \in H \subset \mathcal{F}(P)$  there exists a unique linear transformation  $\Xi : T_{b_0}\mathcal{F}(P) \rightarrow \mathfrak{k}^\perp$  such that

$$d\rho^N(X) = \Xi(X) \cdot \rho^N(b_0). \quad (4.4)$$

The definition of the covariant derivative is

$$\begin{aligned} d\rho^N(X) &= -\psi(X) \cdot \rho^N(b_0) \\ &+ ((\nabla_X n)(b_0), \nabla_X(\nabla n)(b_0), \nabla_X(\nabla^2 n)(b_0), \dots, \nabla_X(\nabla^N n)(b_0)). \end{aligned} \quad (4.5)$$

Under the decomposition  $\mathfrak{so}(n) = \mathfrak{k} \oplus \mathfrak{k}^\perp$  we have  $\psi = \psi^\mathfrak{k} + \psi^{\mathfrak{k}^\perp}$ . Upon restriction to  $H$ ,  $\psi^\mathfrak{k}$  is a  $K$ -connection on the principal bundle  $H$  and  $\psi^{\mathfrak{k}^\perp}$  will become torsion. Since  $K$  is the isotropy group of  $\rho^N(b_0)$  we conclude that  $\psi^\mathfrak{k} \cdot \rho^N(b_0) = 0$ . If we restrict (4.5) to  $H \subset \mathcal{F}(P)$  and choose  $X \in T_{b_0}H$  then  $d\rho^N(X) = 0$  because  $\rho^N$  is constant on  $H$ . Thus for  $X \in T_{b_0}H$  we have that

$$\psi^{\mathfrak{k}^\perp}(X) \cdot \rho^N(b_0) = ((\nabla_X n)(b_0), \nabla_X(\nabla n)(b_0), \nabla_X(\nabla^2 n)(b_0), \dots, \nabla_X(\nabla^N n)(b_0)). \quad (4.6)$$

If we think of the above as a series of linear equations for  $\psi^{\mathfrak{k}^\perp}(X)$  then it is easy to see that if a solution exists then it must be unique. Next we show that the solution exists. To do this we observe that the covariant derivative (with connection  $\psi$ ) of  $\rho^N$  in direction  $e_A$  is given by  $d\rho^N(e_A) = \Xi(e_A) \cdot \rho^N(b_0)$ , see (4.4), (4.5). Thus we have

$$\Xi(e_A) \cdot \rho^N(b_0) = ((\nabla_X n)(b_0), \nabla_X(\nabla n)(b_0), \nabla_X(\nabla^2 n)(b_0), \dots, \nabla_X(\nabla^N n)(b_0)).$$

Comparing this with (4.6) and using the uniqueness of the solution we conclude that the torsion  $\psi^{\mathfrak{k}^\perp}(e_A) = \Xi(e_A)$ . Also note that the right hand side of (4.6) is constant on  $H$  and thus the  $\psi^{\mathfrak{k}^\perp}(e_A)$  must be constant on  $H$  by uniqueness. On restriction to  $H$  we have

$$\psi_{ij}^{\mathfrak{k}^\perp} = \tau_{kij}^{\mathfrak{k}^\perp} \theta^k + \tilde{\tau}_{kij}^{\mathfrak{k}^\perp} \tilde{\theta}^k, \quad (4.7)$$

where  $\tau_{kij}^{\mathfrak{k}^\perp}$  and  $\tilde{\tau}_{kij}^{\mathfrak{k}^\perp}$  are constant on  $H$ .

We are now almost ready to write down the Cartan structural equations for  $H$ . First we observe that certain functions are constant. If we let  $F$  denote  $f_{ijk}$  or  $\tilde{f}_{ijk}$  then equations (I-8.17) and (I-8.18) may schematically be written  $(I+n)F = \nabla n$ . By differentiating we learn that  $\nabla^s F$  is a function of  $(n, \nabla n, \dots, \nabla^{s+1} n)$  only. If  $N \geq 1$  then  $f_{ijk}, \tilde{f}_{ijk}, T''_{ijkl}$  in (4.1), (4.2) and (4.3) are constant on  $H$  since  $\rho^{N+1}$  is constant on  $H$ . We require  $N \geq 1$ . The first Cartan structural equations (4.1) and (4.2) become

$$d\theta^i = -\psi_{ij}^{\mathfrak{k}} \wedge \theta^j - \frac{1}{2} c_{ijk} \theta^j \wedge \theta^k + \tilde{\tau}_{kij}^{\mathfrak{k}^\perp} \theta^j \wedge \tilde{\theta}^k, \quad (4.8)$$

$$d\tilde{\theta}^i = -\psi_{ij}^{\mathfrak{k}} \wedge \tilde{\theta}^j - \frac{1}{2} \tilde{c}_{ijk} \tilde{\theta}^j \wedge \tilde{\theta}^k + \tau_{kij}^{\mathfrak{k}^\perp} \tilde{\theta}^j \wedge \theta^k, \quad (4.9)$$

where  $c, \tilde{c}, \tau^{\mathfrak{k}^\perp}, \tilde{\tau}^{\mathfrak{k}^\perp}$  are constant and  $c_{ijk} = -c_{ikj}$ ,  $\tilde{c}_{ijk} = -\tilde{c}_{ikj}$ . Think of the  $ij$  indices of  $T''_{ijlm}$  as taking values in the  $\mathfrak{so}(n)$  Lie algebra and denote the projection of  $-T''_{ijlm}$  onto  $\mathfrak{k}$  by  $K_{ijlm}^{\mathfrak{k}}$ . Note that  $K_{ijlm}^{\mathfrak{k}}$  is constant on  $H$ . The second Cartan structural equation (4.3) may be written as

$$\begin{aligned} d\psi_{ij}^{\mathfrak{k}} &= -\psi_{ik}^{\mathfrak{k}} \wedge \psi_{kj}^{\mathfrak{k}} - (\psi_{ik}^{\mathfrak{k}} \wedge \psi_{kj}^{\mathfrak{k}^\perp})^{\mathfrak{k}} - (\psi_{ik}^{\mathfrak{k}^\perp} \wedge \psi_{kj}^{\mathfrak{k}})^{\mathfrak{k}} \\ &\quad - (\psi_{ik}^{\mathfrak{k}^\perp} \wedge \psi_{kj}^{\mathfrak{k}^\perp})^{\mathfrak{k}} + K_{ijlm}^{\mathfrak{k}} \theta^l \wedge \tilde{\theta}^m, \end{aligned} \quad (4.10)$$

where you substitute (4.7) for  $\psi^{\mathfrak{k}^\perp}$  in the above. The important lesson is that  $H$  has a coframing given by  $(\theta, \tilde{\theta}, \psi^{\mathfrak{k}})$  and that the structural equations (4.8), (4.9) and (4.10) only involve constants and thus are the Maurer-Cartan equations for a Lie group. We have shown that if  $P$  is infinitesimally  $n$ -homogeneous with  $N \geq 1$  then  $P$  is a homogeneous space  $H/K$  where the Lie group  $H$  is a sub-bundle of the frame bundle  $\mathcal{F}(P)$ .

## 4.2 The case of $K = \{e\}$

This is the situation where the residual symmetry group  $K$  is broken all the way down to the identity group  $\{e\}$ . In this case  $P = H$ , the symplectic manifold  $P$  is a Lie

group, and  $\mathfrak{k}^\perp = \mathfrak{so}(n)$ . The Maurer-Cartan equations for  $P$  are

$$d\theta^i = -\frac{1}{2}c_{ijk}\theta^j \wedge \theta^k + \tilde{\tau}_{kij}\theta^j \wedge \tilde{\theta}^k, \quad (4.11)$$

$$d\tilde{\theta}^i = -\frac{1}{2}\tilde{c}_{ijk}\tilde{\theta}^j \wedge \tilde{\theta}^k + \tau_{kij}\tilde{\theta}^j \wedge \theta^k, \quad (4.12)$$

where  $c_{ijk} = -c_{ikj}$ ,  $\tilde{c}_{ijk} = -\tilde{c}_{ikj}$ . Also  $\tau_{kij}$  and  $\tilde{\tau}_{kij}$  are antisymmetric under  $i \leftrightarrow j$  for arbitrary  $i, j$  reflecting that  $\mathfrak{k}^\perp = \mathfrak{so}(n)$ . Note that  $\tilde{\theta}$  generates a differential ideal and thus  $\tilde{\theta} = 0$  defines a fibration  $\tilde{\Pi} : P \rightarrow \tilde{M}$  with fibers isomorphic to a Lie group  $G$  with structure constants  $c_{ijk}$ . Likewise,  $\theta$  generates a differential ideal and thus  $\theta = 0$  defines a fibration  $\Pi : P \rightarrow M$  with fibers isomorphic to a Lie group  $\tilde{G}$  with structure constants  $\tilde{c}_{ijk}$ . We also remark that  $G$  and  $\tilde{G}$  are Lie subgroups of  $P$ . If  $\tau = 0$  then  $G$  is a normal subgroup of  $P$ . If  $\tilde{\tau} = 0$  then  $\tilde{G}$  is a normal subgroup of  $P$ .

Let  $(e_i, \tilde{e}_j)$  be the basis dual to  $(\theta^i, \tilde{\theta}^j)$ . The Maurer-Cartan equations may be reformulated as the Lie algebra relations

$$[e_i, e_j] = c_{kij}e_k \quad (4.13)$$

$$[\tilde{e}_i, \tilde{e}_j] = \tilde{c}_{kij}\tilde{e}_k \quad (4.14)$$

$$[e_i, \tilde{e}_j] = \tau_{ikj}\tilde{e}_k - \tilde{\tau}_{jki}e_k. \quad (4.15)$$

All the statements made in the previous paragraph also follow from the above.

Consider the left invariant vector fields  $X = X^i e_i$ ,  $Y = Y^i \tilde{e}_i$ . Note that  $\mathcal{L}_X \tilde{\theta}^i = -X^k \tau_{kij} \tilde{\theta}^j$ . Let  $\tau_{ij}(X) = X^k \tau_{kij}$  then using the identity  $[\mathcal{L}_X, \mathcal{L}_Y] \tilde{\theta}^i = \mathcal{L}_{[X, Y]} \tilde{\theta}^i$  you obtain  $[\tau(X), \tau(Y)] = \tau([X, Y])$ . Thus we have a Lie algebra representation  $\tau : \mathfrak{g} \rightarrow \mathfrak{so}(n)$ . This means we have a representation of  $G$  by real orthogonal  $n \times n$  matrices. Likewise,  $\tilde{\tau} : \tilde{\mathfrak{g}} \rightarrow \mathfrak{so}(n)$  is a Lie algebra representation and we have a representation of  $\tilde{G}$  by real orthogonal matrices. This does not mean that  $G$  is a compact group if  $\tau \neq 0$ . A comment made in Appendix A tells us that  $\mathfrak{g}/(\ker \tau)$  is a Lie algebra of the form “compact semisimple + abelian”. We know nothing at all about the ideal  $\ker \tau$  so we cannot make a stronger statement about the structure of  $\mathfrak{g}$ . Similar remarks apply to  $\tilde{G}$ .

A Drinfeld double  $G_D$  admits the following geometric characterization:

1. It is a Lie group of dimension  $2n$  with a bi-invariant quadratic form of type  $(n, n)$ .
2. It is a bifibration with the property that the fibers are isotropic submanifolds of  $G_D$ . The fibers are also isomorphic to Lie groups  $G$  and  $\tilde{G}$ .

If we apply the above to our situation by requiring that the quadratic form  $\tilde{\theta}^i \otimes \theta^i + \theta^i \otimes \tilde{\theta}^i$  be bi-invariant then we learn that  $P = H$  is a Drinfeld double,  $\tau_{ijk} = c_{ijk}$ ,  $\tilde{\tau}_{ijk} = \tilde{c}_{ijk}$ ,  $\tau_{kij} = -\tau_{ikj}$  and  $\tilde{\tau}_{kij} = -\tilde{\tau}_{ikj}$ . It follows that both  $c_{ijk}$  and  $\tilde{c}_{ijk}$  are totally antisymmetric and thus the quadratic forms  $\theta^i \otimes \theta^i$  and  $\tilde{\theta}^i \otimes \tilde{\theta}^i$  give bi-invariant positive definite metrics on  $G$  and  $\tilde{G}$  respectively. Thus  $G$  and  $\tilde{G}$  are of the type “compact semisimple + abelian”. The symplectic structure on  $P$  (if it exists) appears to be different than the standard symplectic structure on the Drinfeld double, see (3.1). In the examples I am familiar, if  $G$  is simple and compact the its dual  $\tilde{G}$  constructed via  $R$ -matrices is neither simple nor compact. I do not know what is known in this more general case “compact semisimple + abelian”.

Returning to the general case we remark that the equations  $d^2\theta = 0$  and  $d^2\tilde{\theta} = 0$  leads to the conclusion that  $c_{ijk}$  and  $\tilde{c}_{ijk}$  are respectively the structure constants for Lie groups  $G$  and  $\tilde{G}$ ,  $\tau : \mathfrak{g} \rightarrow \mathfrak{so}(n)$  and  $\tilde{\tau} : \tilde{\mathfrak{g}} \rightarrow \mathfrak{so}(n)$  are Lie algebra representations and

$$\tau_{kjm}\tilde{\tau}_{jil} - \tau_{ljm}\tilde{\tau}_{jik} + c_{ijk}\tilde{\tau}_{mjl} - c_{ijl}\tilde{\tau}_{mjk} + c_{jkl}\tilde{\tau}_{mij} = 0, \quad (4.16)$$

$$\tilde{\tau}_{kjm}\tau_{jil} - \tilde{\tau}_{ljm}\tau_{jik} + \tilde{c}_{ijk}\tau_{mjl} - \tilde{c}_{ijl}\tau_{mjk} + \tilde{c}_{jkl}\tau_{mij} = 0. \quad (4.17)$$

These equations are generalizations of the corresponding equations (B.4) in the bialgebra case. These equations follow just from the structure equations for the group  $P = H$ . There are additional constraints which follow from duality considerations such as  $d\gamma = H - \tilde{H}$  which lead to

$$c_{ijk}n_{il} + c_{ikl}n_{ij} + c_{ilj}n_{ik} = +H_{jkl}, \quad (4.18)$$

$$\tilde{c}_{ijk}n_{il} + \tilde{c}_{ikl}n_{ij} + \tilde{c}_{ilj}n_{ik} = -\tilde{H}_{jkl}, \quad (4.19)$$

$$m_{jl}\tau_{kli} - m_{kl}\tau_{jli} + n_{jl}\tilde{\tau}_{ilk} - n_{kl}\tilde{\tau}_{ilj} = -m_{il}c_{ljk}, \quad (4.20)$$

$$m_{jl}\tilde{\tau}_{kli} - m_{kl}\tilde{\tau}_{jli} + n_{jl}\tau_{ilk} - n_{kl}\tau_{ilj} = +\tilde{c}_{ljk}m_{li}. \quad (4.21)$$

We do not know if there are non-trivial solutions to these equations.

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## Appendices

### A Some Lie groups facts

For clarification purposes we make some remarks about the left and right actions on a Lie group. For notational simplicity we restrict to matrix Lie groups. The identity element will be denoted by  $I$ . Let  $G$  be a Lie group. Let  $a \in G$  then the left and right actions on  $G$  are respectively defined by  $L_a g = ag$  and  $R_a g = ga$  for  $g \in G$ . We take  $\mathfrak{g}$ , the Lie algebra of  $G$ , to be the left invariant vector fields and we identify it with the tangent space at the identity  $T_I G$ . The left invariant Maurer-Cartan forms are  $\theta = g^{-1}dg$ . They satisfy the Maurer-Cartan equations  $d\theta = -\theta \wedge \theta$ . Pick a basis  $(e_1, \dots, e_n)$  of left invariant vector fields for  $\mathfrak{g}$  with bracket relations  $[e_i, e_j] = f^k_{ij} e_k$ . If the dual basis of left invariant forms is  $(\theta^1, \dots, \theta^n)$  then  $d\theta^i = -\frac{1}{2}f^i_{jk}\theta^j \wedge \theta^k$ .

Naively you would expect that if  $X$  is a left invariant vector field then  $\mathcal{L}_X \theta = 0$  since  $\theta$  is left invariant. In fact a brief computation shows that  $\mathcal{L}_X \theta^i = -(X^k f^i_{kj})\theta^j$  which is the adjoint action. The answer to this conundrum is that the left invariant vector fields generate the right group action. The easiest way to see this is to use old fashioned differentials. Let  $v \in T_I G$  then the left invariant vector field  $X$  at  $g$  which is  $v$  at the identity is given by  $X = gv$ . The infinitesimal action of this vector field at  $g$  is given by  $g \rightarrow g + \epsilon X = g + \epsilon gv = g(I + \epsilon v) \approx g e^{\epsilon v}$  which is the right action of the group. Thus we see that  $g^{-1}dg \rightarrow e^{-\epsilon v}(g^{-1}dg)e^{\epsilon v}$  which is the adjoint action in accordance with the Lie derivative computation. Take any metric  $h_{ij}$  at the identity then  $h_{ij}\theta^i \otimes \theta^j$  is a left invariant metric on  $G$ . In general this metric is not invariant under the right action of the group. The right invariance condition is  $h_{il}f^l_{jk} + h_{jl}f^l_{ik} = 0$  which means that the structure constants are totally antisymmetric if the indices are lowered using  $h_{ij}$ . In such a situation the metric is bi-invariant.

Assume you have a Lie algebra  $\mathfrak{g}$  with an invariant positive definite metric then you have an orthogonal decomposition into ideals  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a}$  where  $\mathfrak{k}$  is a compact semisimple Lie algebra and  $\mathfrak{a}$  is abelian. The proof is straightforward and involves putting together a variety of observations. The invariance of the inner product  $(\cdot, \cdot)$  is the statement that the adjoint representation  $\text{ad}_X Y = [X, Y]$  is skew adjoint with respect to the metric:  $(\text{ad}_X Y, Z) + (Y, \text{ad}_X Z) = 0$ . The skew adjointness immediately leads to the

decomposition of  $\mathfrak{g}$  into irreducible pieces. It is an elementary exercise in linear algebra to show that if  $\mathfrak{h}$  is a non-trivial ideal in  $\mathfrak{g}$  then its orthogonal complement  $\mathfrak{h}^\perp$  is also an ideal. Since  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$  we conclude that by continuing this process the Lie algebra decomposes into irreducible ideals  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_N$ . Let us collate all the abelian subalgebras together into  $\mathfrak{a}$  and rewrite the decomposition as  $\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{k}_2 \oplus \cdots \oplus \mathfrak{k}_M \oplus \mathfrak{a}$  where each  $\mathfrak{k}_j$  is a simple lie algebra. Since this is a decomposition into ideals we have that  $\mathfrak{a}$  is the center of the Lie algebra. Let  $(\cdot, \cdot)_j$  be the restriction of the invariant inner product to  $\mathfrak{k}_j$ . An application of Schur's Lemma tells us that an invariant bilinear form on a simple Lie algebra is a multiple of the Killing form. Thus we conclude that  $(\cdot, \cdot)_j = \lambda K_j(\cdot, \cdot)$  where  $\lambda$  is a non-zero scalar and  $K_j$  is the Killing form<sup>1</sup> on  $\mathfrak{k}_j$ . Since  $(\cdot, \cdot)$  is positive definite, the Killing form  $K_j$  must be definite and this is only possible if the Lie algebra  $\mathfrak{k}_j$  is of compact type. This concludes the proof of the proposition in the opening sentence.

Closely related to the above is the following. If a Lie algebra  $\mathfrak{g}$  has a faithful representation  $\tau : \mathfrak{g} \rightarrow \mathfrak{so}(n)$  by skew adjoint matrices then  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a}$  where  $\mathfrak{k}$  is a compact semisimple Lie algebra and  $\mathfrak{a}$  is abelian. The proof follows from the observation that because the representation is faithful we can think of  $\mathfrak{g}$  as a matrix Lie subalgebra of  $\mathfrak{so}(n)$ . We know that  $\mathfrak{so}(n)$  has a positive definite invariant metric so restriction to  $\mathfrak{g}$  induces a positive definite invariant metric on  $\mathfrak{g}$ . We now use the proposition from the previous paragraph.

We remark that if the representation  $\tau$  in the previous paragraph is not faithful then  $(\ker \tau) \subset \mathfrak{g}$  is a nontrivial ideal in  $\mathfrak{g}$ . The Lie algebra  $\mathfrak{g}/(\ker \tau)$  is of the form  $\mathfrak{k} \oplus \mathfrak{a}$  but we can say nothing about the Lie algebra  $(\ker \tau)$ .

It is easy to see that the space of all invariant positive definite metrics on a Lie algebra is a convex set. In a simple Lie algebra  $\mathfrak{g}$ , the third cohomology group  $H^3(\mathfrak{g})$  is one dimensional and generated by the three form  $\omega(X, Y, Z) = K(X, [Y, Z])$  where  $K$  is the killing form which may be written in terms of the structure constants as  $\omega_{ijk} = K_{il} f^l_{jk}$ . If  $h$  is an invariant metric then  $h(X, [Y, Z])$  is also a closed three form and by the convexity of the space of positive definite invariant metrics it must be in the same cohomology class as  $\omega$ .

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<sup>1</sup>The sign of the Killing form is chosen such that it is positive on a compact simple Lie algebra.

## B A primer on classical $R$ -matrices

There are two main nonequivalent approaches to classical  $R$ -matrices. The more familiar one is due to Drinfeld and based on the study of Lie bialgebras [16]. The other due to Semenov-Tian-Shansky is based on double Lie algebras [15] is the one directly related to our work. Here we discuss the interconnections between these two approaches. For an introduction to  $R$ -matrices and Poisson-Lie groups see the book by Chari and Pressley [21] or the article [19].

### B.1 The Drinfeld Approach

Drinfeld bases his approach on the notion of a Lie bialgebra. We begin with a down to earth approach. Assume you have a Lie algebra  $\mathfrak{g}$  with basis  $(e_1, \dots, e_n)$  and Lie bracket relations  $[e_a, e_b] = f^c_{ab} e_c$ . If  $X, Y \in \mathfrak{g}$  then the adjoint action by  $X$  is given by  $\text{ad}_X Y = [X, Y]$ . The adjoint action extends naturally to tensor products of  $\mathfrak{g}$ . Let  $\mathfrak{g}^*$  be the vector space dual with corresponding basis  $(\omega^1, \dots, \omega^n)$ . The Lie algebra  $\mathfrak{g}$  acts on  $\mathfrak{g}^*$  via the coadjoint representation  $\text{ad}^*_{e_a} \omega^b = -f^b_{ac} \omega^c$ . In general  $\mathfrak{g}^*$  is not a Lie algebra but there is a natural Lie algebra structure on  $\mathfrak{g} \oplus \mathfrak{g}^*$  given by

$$\begin{aligned} [e_a, e_b] &= f^c_{ab} e_c, \\ [e_a, \omega^b] &= -f^b_{ac} \omega^c \\ [\omega^a, \omega^b] &= 0. \end{aligned}$$

The most famous example is combining  $\mathfrak{g} = \mathfrak{so}(3)$  and its Lie algebra dual  $\mathfrak{g}^* \approx \mathbb{R}^3$  into the Lie algebra of the euclidean group  $E(3)$ . The situation becomes much more interesting when  $\mathfrak{g}^*$  is a Lie algebra in its own rights  $[\omega^a, \omega^b] = \hat{f}^c_{ab} \omega^c$ . You observe that  $\mathfrak{g}^*$  acts via its coadjoint action on its dual  $(\mathfrak{g}^*)^* \approx \mathfrak{g}$ . Thus one can consider the following more symmetric structure which takes into account the respective coadjoint actions

$$[e_a, e_b] = f^c_{ab} e_c, \tag{B.1}$$

$$[e_a, \omega^b] = \hat{f}^b_{ac} e_c - f^b_{ac} \omega^c, \tag{B.2}$$

$$[\omega^a, \omega^b] = \hat{f}^c_{ab} \omega^c. \tag{B.3}$$

This will be a Lie algebra if the following conditions are satisfied:

$$f^e_{ab} \hat{f}^{cd} = \hat{f}^{ed} f^c_{ae} + \hat{f}^{ce} f^d_{ae} - \hat{f}^{ed} f^c_{be} - \hat{f}^{ce} f^d_{be}. \tag{B.4}$$



According to Drinfeld a *Lie bialgebra* is a Lie algebra with Lie brackets (B.1), (B.2) and (B.3).

Drinfeld gives a more abstract formulation which is more suitable for studying the abstract properties of a bialgebra and seeing the origins of classical  $R$ -matrices. Assume you have a Lie algebra  $\mathfrak{g}$  and a “cobracket”  $\Delta : \mathfrak{g} \rightarrow \bigwedge^2 \mathfrak{g}$ . Drinfeld requires that the cobracket defines a Lie algebra on  $\mathfrak{g}^*$ . The structure constants on  $\mathfrak{g}^*$  are related to the cobracket by  $\Delta(e_a) = \frac{1}{2} \hat{f}_a^{bc} e_b \wedge e_c$ . Compatibility condition (B.4) is incorporated via a cohomological argument. The complex in question is

$$\bigwedge^2 \mathfrak{g} \longrightarrow \mathfrak{g}^* \otimes \bigwedge^2 \mathfrak{g} \xrightarrow{\partial^*} \bigwedge^2 \mathfrak{g}^* \otimes \bigwedge^2 \mathfrak{g} \longrightarrow \cdots .$$

The coboundary operator (differential) is given by

$$(\partial^* \Delta)(X, Y) = \text{ad}_X(\Delta(Y)) - \text{ad}_Y(\Delta(X)) - \Delta([X, Y]) . \quad (\text{B.5})$$

The condition (B.4) that glues the Lie algebras into a Lie bialgebra is seen to be equivalent to the cocycle condition  $\partial^* \Delta = 0$ . In this language, a *Lie bialgebra* is a Lie algebra  $\mathfrak{g}$  along with a cobracket  $\Delta$  such that  $\mathfrak{g}^*$  a Lie algebra and the cobracket is a 1-cocycle.

A Lie bialgebra is exact if the cocycle is exact. This means that there exists a  $r \in \bigwedge^2 \mathfrak{g}$  such that  $\Delta(X) = \partial^* r = \text{ad}_X r$ . A computation shows that  $r$  defines a bialgebra structure if and only if the Schouten bracket  $\llbracket r, r \rrbracket$  is  $\text{ad}(\mathfrak{g})$ -invariant:  $\llbracket r, r \rrbracket \in (\bigwedge^3 \mathfrak{g})_{\text{inv}}$ . The Schouten bracket is defined by

$$\llbracket W \wedge X, Y \wedge Z \rrbracket = [W, Y] \wedge X \wedge Z - [W, Z] \wedge X \wedge Y - [X, Y] \wedge W \wedge Z + [X, Z] \wedge W \wedge Y.$$

The condition  $\llbracket r, r \rrbracket \in (\bigwedge^3 \mathfrak{g})_{\text{inv}}$  is called the modified classical Yang-Baxter equation (MCYBE) and  $\llbracket r, r \rrbracket = 0$  is called the classical Yang-Baxter equation (CYBE).

If  $\mathfrak{g}$  is semisimple then the Whitehead lemma states that  $H^1(\mathfrak{g}, \bigwedge^2 \mathfrak{g}) = 0$  and thus the cocycle  $\Delta$  is always a coboundary  $\Delta = \partial^* r$ . Thus in this case we need to understand the set of all  $r \in \bigwedge^2 \mathfrak{g}$  which satisfy MCYBE.

If  $\mathfrak{g}$  is simple then  $(\bigwedge^3 \mathfrak{g})_{\text{inv}}$  is one dimensional and is generated by the three index tensor obtained by raising two indices on the structure constants using the Killing metric. If we call this object  $B_K$  then MCYBE becomes  $\llbracket r, r \rrbracket = a B_K$  for some constant  $a$ .

Let us work in a basis. If  $r = \frac{1}{2} r^{ab} e_a \wedge e_b$ . Then  $\Delta(e_a) = \text{ad}_{e_a} r = \frac{1}{2} r^{bc} \text{ad}_{e_a}(e_b \wedge e_c) = \frac{1}{2} (r^{dc} f_{ad}^b + r^{bd} f_{ad}^c) e_b$ . This tells us that

$$\hat{f}_a^{bc} = r^{dc} f_{ad}^b + r^{bd} f_{ad}^c . \quad (\text{B.6})$$

Note that  $\hat{f}_a^{bc} = -\hat{f}_a^{cb}$  because  $r$  is skew.

It is possible to generalize the above by allowing the cocycle (now called  $C$ ) to be in  $\mathfrak{g} \otimes \mathfrak{g}$ . If you write  $C = C^{ab}e_a \otimes e_b$  and you let  $\Delta(X) = \text{ad}_X C$ . In this case you find you get a Lie bialgebra if the following two conditions are satisfied:

1.  $(C^{ab} + C^{ba})e_a \otimes e_b$ , the symmetric part of  $C$ , is  $\text{ad}(\mathfrak{g})$ -invariant,
2.  $\llbracket C, C \rrbracket \equiv [C^{12}, C^{13}] = [C^{12}, C^{23}] + [C^{13}, C^{23}] \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$  is  $\text{ad}(\mathfrak{g})$ -invariant.

We use standard quantum group notation where for example  $C^{13} = C^{ab}e_a \otimes I \otimes e_b$ , *etc.* The bracket  $\llbracket \cdot, \cdot \rrbracket$  above reduces to the Schouten bracket if  $C$  is skew symmetric. We remark that the equation  $\llbracket C, C \rrbracket = 0$  is also called the classical Yang-Baxter equation.  $C$  or  $r$  are called classical  $R$ -matrices (by Drinfeld). The modified classical Yang-Baxter equation is  $\llbracket C, C \rrbracket \in (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})_{\text{inv}}$ .

A brief computation shows that

$$\hat{f}_a^{bc} = C^{dc}f_{ad}^b + C^{bd}f_{ad}^c. \quad (\text{B.7})$$

Let us write  $C = \frac{1}{2}s^{ab}(e_a \otimes e_b + e_b \otimes e_a) + \frac{1}{2}r^{ab}e_a \wedge e_b$  where  $s$  is symmetric and  $r$  is antisymmetric. Since  $s^{ab}$  is an  $\text{ad}(\mathfrak{g})$ -invariant tensor we have that

$$\hat{f}_a^{bc} = r^{dc}f_{ad}^b + r^{bd}f_{ad}^c. \quad (\text{B.8})$$

Thus the Lie algebra structure on  $\mathfrak{g}^*$  only depends on the antisymmetric part of  $C$ . Note that  $\hat{f}_a^{bc}$  will be skew under  $b \leftrightarrow c$  as required. The effect of the symmetric part  $s_{ab}$  is to change the  $\text{ad} \mathfrak{g}$ -invariant term in the right hand side of the MCYBE. Equation (B.7) may be interpreted as giving a Lie algebra homomorphism  $C : \mathfrak{g}^* \rightarrow \mathfrak{g}$ .

## B.2 Semenov-Tian-Shansky approach

Semenov-Tian-Shansky's approach is directly influenced by classical integrable models where he needs that a single Lie algebra admits two different Lie brackets. Let  $\mathfrak{g}$  be a Lie algebra and let  $R : \mathfrak{g} \rightarrow \mathfrak{g}$  be a linear transformation (not necessarily invertible). Define a skew operation  $[\cdot, \cdot]_R$  by

$$[X, Y]_R = [RX, Y] + [X, RY]. \quad (\text{B.9})$$

If  $[\cdot, \cdot]_R$  is a Lie bracket then  $R$  is called a classical  $R$ -matrix by Semenov-Tian-Shansky. The Jacobi identity for  $[\cdot, \cdot]_R$  may be written as

$$\left[ X, [RY, RZ] - R([Y, Z]_R) \right] + \text{cyclic permutations} = 0$$

A Lie algebra  $\mathfrak{g}$  with two Lie brackets  $[\cdot, \cdot]$  and  $[\cdot, \cdot]_R$  is called a *double Lie algebra* by Semenov-Tian-Shansky. The equation is  $[RY, RZ] - R([Y, Z]_R) = 0$  is also called the CYBE. The equation

$$[RY, RZ] - R([Y, Z]_R) = -c[Y, Z] \quad (\text{B.10})$$

where  $c$  is a constant is also called the MCYBE. Solutions to either of these satisfy the Jacobi identity displayed above.

In a basis we have that  $Re_a = e_b R^b_a$ ,  $[e_a, e_b]_R = (f^c_{ab})_R e_c$  and consequently (B.9) becomes  $[e_a, e_b]_R = (R^d_a f^c_{db} + R^d_b f^c_{ad}) e_c$ . Thus the new structure constants are

$$(f^c_{ab})_R = R^d_a f^c_{db} + R^d_b f^c_{ad}. \quad (\text{B.11})$$

The Semenov-Tian-Shansky version of the MCYBE (B.10) is

$$f^c_{de} R^d_a R^e_b - f^e_{db} R^d_a R^c_e - f^e_{ad} R^d_b R^c_e = -c f^c_{ab}. \quad (\text{B.12})$$

### B.3 Relating Drinfeld and Semenov-Tian-Shansky

To relate the Semenov-Tian-Shansky approach and the Drinfeld approach one needs an  $\text{ad}(\mathfrak{g})$ -invariant metric on  $\mathfrak{g}$ . If  $\mathfrak{g}$  is semisimple then one can take the  $\text{ad}(\mathfrak{g})$ -invariant metric to be the Killing metric. The  $\text{ad}(\mathfrak{g})$ -invariant metric is used to identify  $\mathfrak{g}$  with  $\mathfrak{g}^*$ . By lowering indices  $f_{abc}$  is completely antisymmetric (due to the  $\text{ad}(\mathfrak{g})$ -invariance). We wish to identify  $f_R$  with  $\tilde{f}$ . Note that by rearranging indices we have

$$\begin{aligned} (f^a_b)_R &= R^{da} f_{cd}^b + R^{db} f_c^a{}_d \\ &= R^{da} f^b_{cd} - R^{db} f^a_{cd} \end{aligned} \quad (\text{B.13})$$

If we are in a situation where  $R^{ab} = -r^{ab}$  then we have related  $f_R$  to  $\hat{f}$ . Said differently  $R : \mathfrak{g} \rightarrow \mathfrak{g}$  is a skew-adjoint operator with respect to the invariant metric. In fact there is a theorem [15] which states that if  $\mathfrak{g}$  has an  $\text{ad}(\mathfrak{g})$ -invariant metric and if  $R : \mathfrak{g} \rightarrow \mathfrak{g}$  is skew-adjoint then the double Lie algebra is isomorphic to a Lie bialgebra and all the structures in the Semenov-Tian-Shansky approach (CYBE, MYBE) go into the structures in the Drinfeld approach (CYBE, MYBE). The isomorphism is given by thinking of the metric as giving a map  $\mathfrak{g} \rightarrow \mathfrak{g}^*$ , *i.e.*, lowering/raising indices. We are in a different situation because not all our  $R$ -matrices are skew adjoint.

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